The zero-momentum limit of thermal green functions

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Abstract

The zero momentum limit of thermal self-energies calculated in perturbation theory depends on the order in which the time and the space components of the momentum are taken to zero. We show that this is an artifact of the perturbative calculation, and in fact the limit is well-defined when higher orders of the perturbation expansion are properly taken into account.

The existing calculations of thermal self-energy functions using the formalism of Thermal Field Theory yield results that are not defined if the external momentum 4-vector is zero. The classic example is the photon self-energy $\pi_{\mu\nu}$ in an electron gas. It is well known [1] that the result of the one-loop calculation of $\pi_{\mu\nu}(p^0,\vec{p})$ for a photon with external momentum $p^\mu = (p^0,\vec{p})$ is such that

$$\lim_{|\vec{p}|\to 0} \pi_{\mu\nu}(0,\vec{p}) \neq \lim_{p^0\to 0} \pi_{\mu\nu}(p^0,\vec{0})\,, \tag{1}$$

so that the limit in which all components go to zero is not defined. To be more specific, the above-mentioned problem occurs for the real part of the self-energy, while the imaginary part is well defined.

Various attempts have been made to resolve this puzzle [2, 3, 4, 5], which involve either introducing new and ad-hoc Feynman rules for thermal field theories, or putting restrictions on the general rules. Along another line of attempt, it has recently been pointed out by Arnold, Vokos, Bedaque and Das (AVBD) [6] that the problem mentioned above occurs only if the self energy diagram contains two propagators of the same mass. If the masses of the particles in the loop are different, the problem does not exist. In fact, the calculations of the neutrino self-energy in a gas of electrons and nucleons, which were carried out even before the work of Ref. [6], show this [7, 8, 9]. It has been speculated that this property may be utilized to introduce a mass-splitting regularization for thermal diagrams [6] in cases where problems are known to occur.

This problem, as well the attempts to resolve it, are based on the results of one-loop perturbative calculations. It is natural to ask whether the singular behaviour of the self-energy function at zero momentum might be a consequence of the approximations and idealizations that are implicitly made in the perturbative calculations. In this article we show that this is precisely the case, and that the calculation of the self-energy beyond one-loop order yields a result that is defined at zero momentum. This result is obtained by calculating the self-energies using the full propagators for the particles that appear in the internal lines of the loop diagrams. Some of these propagators have an absorptive part which, as observed by AVBD [6], is well-defined at zero momentum even if they are calculated to one-loop and the particles in the internal lines of the loop have the same mass. As we will see, this in turn governs the zero-momentum limit of those self-energy diagrams in which the internal lines have the same mass. We exemplify these assertions by calculating the photon self-energy in scalar QED, but similar considerations apply to QED proper as well.

Before proceeding, we recapitulate some results of the canonical approach to the thermal propagators, which we will be using throughout [10]. In this approach, one has to use anti-time-ordered propagators in addition to the time-ordered ones, as well as propagators with no time-ordering. These propagators can be arranged in the form of a 2×2 matrix. For example, for any bosonic field Φ^A where A denotes any Lorentz index carried by the field

(none for a scalar field, one for the photon), we can write

$$i\boldsymbol{\mathcal{D}}_{11}^{AB}(x-y) \equiv \left\langle \mathcal{T}\ \Phi^A(x)\Phi^{B*}(y)\right\rangle , \tag{2}$$
$$i\boldsymbol{\mathcal{D}}_{22}^{AB}(x-y) \equiv \left\langle \overline{\mathcal{T}}\ \Phi^A(x)\Phi^{B*}(y)\right\rangle , \tag{3}$$
$$i\boldsymbol{\mathcal{D}}_{12}^{AB}(x-y) \equiv \left\langle \Phi^{B*}(y)\Phi^A(x)\right\rangle , \tag{4}$$
$$i\boldsymbol{\mathcal{D}}_{21}^{AB}(x-y) \equiv \left\langle \Phi^A(x)\Phi^{B*}(y)\right\rangle , \tag{5}$$

where the time-ordering and the anti-time-ordering operators $\mathcal{T}$ and $\overline{\mathcal{T}}$ are defined as

$$\mathcal{T}\ \Phi^A(x)\Phi^{B*}(y) \equiv \Theta(x_0-y_0)\Phi^A(x)\Phi^{B*}(y)+\Theta(y_0-x_0)\Phi^{B*}(y)\Phi^A(x) , \tag{6}$$
$$\overline{\mathcal{T}}\ \Phi^A(x)\Phi^{B*}(y) \equiv \Theta(y_0-x_0)\Phi^A(x)\Phi^{B*}(y)+\Theta(x_0-y_0)\Phi^{B*}(y)\Phi^A(x) , \tag{7}$$

Θ being the step function. We can now make the momentum space expansion of the field as

$$\Phi^A(x)=\int\frac{d^3p}{(2\pi)^3 2E}\sum_\lambda\left[a_\lambda(p)u^A(p,\lambda)e^{-ip\cdot x}+b_\lambda^*(p)v^A(p,\lambda)e^{ip\cdot x}\right] , \tag{8}$$

where u^A and v^A represent different plane wave solutions arranged by the index λ, and $a_\lambda(p)$ and $b_\lambda(p)$ are the annihilation operators for particles and antiparticles, respectively. For a self-adjoint field like the photon, $a_\lambda(p) = b_\lambda(p)$. The properties of the thermal bath come in from the expectation values

$$\langle a_\lambda(p)a_{\lambda'}^*(p')\rangle = (2\pi)^3 2E\delta(\vec{p}-\vec{p}\,')\delta_{\lambda\lambda'}\left[1+f_B(p,\alpha)\right] , \tag{9}$$
$$\langle b_\lambda(p)b_{\lambda'}^*(p')\rangle = (2\pi)^3 2E\delta(\vec{p}-\vec{p}\,')\delta_{\lambda\lambda'}\left[1+f_B(p,-\alpha)\right] , \tag{10}$$

with

$$f_B(p,\alpha)=\frac{1}{e^{\beta p\cdot u-\alpha}-1} , \tag{11}$$

where α plays the role of a chemical potential. We have introduced the velocity 4-vector u^μ of the heat bath, which has components $(1,\vec{0})$ in its own rest frame.

For scalar fields, the procedure described above gives [10] the 2×2 free-field propagator as

$$\boldsymbol{\Delta}(p) = \boldsymbol{\mathcal{U}}_B \begin{pmatrix} \Delta_0 & 0 \\ 0 & -\Delta_0^* \end{pmatrix} \boldsymbol{\mathcal{U}}_B \,, \tag{12}$$

where

$$\Delta_0 \equiv \frac{1}{p^2 - m^2 + i0} \,, \tag{13}$$

and

$$\boldsymbol{\mathcal{U}}_B = \frac{1}{\sqrt{1+\eta_B(p,\alpha)}} \begin{pmatrix} 1+\eta_B(p,\alpha) & \epsilon(p\cdot u) f_B(p,\alpha) \\ -\epsilon(p\cdot u) f_B(-p,-\alpha) & 1+\eta_B(p,\alpha) \end{pmatrix} \,, \tag{14}$$

with $\epsilon(x) \equiv \Theta(x) - \Theta(-x)$, and

$$\eta_B(p,\alpha) = \Theta(p\cdot u) f_B(p,\alpha) + \Theta(-p\cdot u) f_B(-p,-\alpha) \,. \tag{15}$$

It is straightforward to check that this gives, for example,

$$\boldsymbol{\Delta}_{11}(p) \;=\; \frac{1}{p^2 - m^2 + i0} - 2\pi i \delta(p^2 - m^2)\eta_B(p,\alpha) \,, \tag{16}$$

which is the propagator given by Dolan and Jackiw [11]. The explicit forms of the other components are also given in the literature [12].

The one-loop diagrams for the photon self-energy in a background of ϕ particles are depicted in Fig. 1. Diagram 1b produces a term that is proportional to the total electric charge of the system. However, since that term is a constant, independent of the photon momentum, it will not be relevant for our discussion and we will not consider it any further.

The result of calculating Diagram 1a using the free-field propagator given above for the ϕ field is

$$\begin{aligned} \mathrm{Re}\,[\boldsymbol{\pi}_{\mu\nu}(k_0,\vec{k})]_{11} \;=\;& e^2 \int \frac{d^4p}{(2\pi)^3} \eta_B(p,\alpha)\delta(p^2 - m^2) \\ & \times \left\{ \frac{(2p+k)_\mu (2p+k)_\nu}{k^2 + 2p\cdot k} + (k \to -k) \right\} \end{aligned} \tag{17}$$

where the vacuum contribution has been omitted, as we will always do henceforth whenever we write explicit expressions for the self-energies. The above formula reveals the problem to which we alluded in Eq. (1). As we mentioned earlier, this problem vanishes if the diagram is evaluated employing the full propagator of the ϕ field instead of the free-field propagator. The full scalar propagator, which we denote by $\mathbf{\Delta}'(p)$, can be written just like in Eq. (12) but with Δ_0 replaced by

$$\Delta_0' = \frac{1}{p^2 - m^2 - \Pi_0}, \tag{18}$$

where Π_0 is the self-energy function for the ϕ field. Thus,

$$\mathbf{\Delta}'(p) = \boldsymbol{\mathcal{U}}_B \left(\begin{array}{cc} \Delta_0' & 0 \\ 0 & -\Delta_0'^* \end{array} \right) \boldsymbol{\mathcal{U}}_B \,, \tag{19}$$

and in particular,

$$\mathbf{\Delta}'_{11}(p) = \Delta_0' + [\Delta_0' - \Delta_0'^*]\, \eta_B(p,\alpha)\,. \tag{20}$$

The important consequence of replacing Δ_0 by Δ_0' is that the δ-function present in Eq. (16) is now smeared if the absorptive part of Π_0 is non-zero. For this reason it is the easy to see that, if Diagram 1a is calculated with the propagator $\mathbf{\Delta}'_{11}$ for the ϕ field instead of the free particle propagator, then the problem at zero momentum does not arise in evaluating $[\boldsymbol{\pi}_{\mu\nu}(k)]_{11}$. Notice that the dispersive part of Π_0 plays no role in this argument. It is not difficult to see that retaining only the dispersive part of Π_0 and neglecting its absorptive part does not remove the singularity of $[\boldsymbol{\pi}_{\mu\nu}]_{11}$ at zero momentum.

The next step is to calculate Π_0 and show that in general it has an absorptive part. To this end, we recall that the inverse of the full scalar propagator is given by

$$\mathbf{\Delta}'^{-1}(p) = p^2 - m^2 - \mathbf{\Pi}\,, \tag{21}$$

where $\mathbf{\Pi}$ is a 2×2 matrix whose components must be calculated using the Feynman rules of the theory. Comparing Eqs. (19) and (21), the following relations are obtained,

$$\begin{aligned}
\mathbf{\Pi}_{11} &= \Pi_0 + (\Pi_0 - \Pi_0^*)\eta_B(p,\alpha) && (22)\\
\mathbf{\Pi}_{22} &= -\Pi_0^* + (\Pi_0 - \Pi_0^*)\eta_B(p,\alpha) && (23)\\
\mathbf{\Pi}_{12} &= -(\Pi_0 - \Pi_0^*)\epsilon(p\cdot u) f_B(p,\alpha) && (24)\\
\mathbf{\Pi}_{21} &= -(\Pi_0 - \Pi_0^*)\epsilon(-p\cdot u) f_B(-p,-\alpha)\,. && (25)
\end{aligned}$$

from which it is easily seen that

$$\mathrm{Re}\,\Pi_0(p) = \mathrm{Re}\,\mathbf{\Pi}_{11}(p) \tag{26}$$

$$\mathrm{Im}\,\Pi_0(p) = \frac{\epsilon(p\cdot u)\mathbf{\Pi}_{12}(p)}{2if_B(p,\alpha)}\,. \tag{27}$$

Therefore, to determine Π_0 we must calculate $\mathbf{\Pi}_{11}$ and $\mathbf{\Pi}_{12}$, which can be done by evaluating the diagrams in Fig. 2. Since the dispersive part Π_0 is not relevant for resolving the zero momentum problem of the photon self-energy, we will not calculate it here. However, for the consistency of our scheme it is important to stress that since the internal lines in Fig. 2 correspond to particles of different mass, the function $\mathrm{Re}\,\mathbf{\Pi}_{11}$ (and hence $\mathrm{Re}\,\Pi_0$) does not suffer from the zero momentum problem according to the observation of AVBD [6].

We now turn the attention to the calculation of the absorptive part of Π_0. The simplest way to proceed is to calculate $\mathbf{\Pi}_{12}$ and then use Eq. (27). As in the case of Diagram 1b for the photon self-energy, the diagram in Fig. 2b is irrelevant for our purpose. In Fig. 2a, the scalar propagator that enters for $\mathbf{\Pi}_{12}$ is obtained from Eq. (12) as

$$\mathbf{\Delta}_{12}(p') = -2\pi i\delta(p'^2 - m^2)f_B(p',\alpha)\epsilon(p'\cdot u)\,. \tag{28}$$

In addition to this, we also need the thermal photon propagator, whose form depends on the gauge that is chosen. Here we will use the Coulomb gauge which, for several reasons, has been advocated as a convenient one for carrying out finite temperature calculations in QED [13, 14]. The only component that is needed for the calculation at hand is the 21 component which, borrowing from Refs. [13, 14], is given by

$$\boldsymbol{D}_{21}^{\mu\nu}(k) = 2\pi i\delta(k^2)f_B(-k,0)\epsilon(k\cdot u)(-S^{\mu\nu})\,, \tag{29}$$

where

$$S_{\mu\nu} = g_{\mu\nu} + \frac{1}{\kappa^2}k_\mu k_\nu - \frac{\omega}{\kappa^2}(u_\mu k_\nu + k_\mu u_\nu)\,. \tag{30}$$

Here ω and κ are defined by

$$\omega = k\cdot u\,, \qquad \kappa = \sqrt{\omega^2 - k^2}\,, \tag{31}$$

and they represent the energy and momentum of the photon in the rest frame of the heat bath. A particularly useful relation is

$$\sum_{\lambda=1,2} \epsilon_\mu(k,\lambda)\epsilon_\nu(k,\lambda)\Big|_{\omega=\kappa} = -\, S_{\mu\nu}\Big|_{\omega=\kappa} \,, \tag{32}$$

where the polarization vectors $\epsilon_\mu(k,\lambda) = (0, \vec{e}(k,\lambda))$ are such that, in the rest frame of the heat bath,

$$\vec{k}\cdot\vec{e}(k,1) = \vec{k}\cdot\vec{e}(k,2) = 0\,. \tag{33}$$

The application of the Feynman rules to the diagram of Fig. 2a gives

$$-\, i\mathbf{\Pi}_{12}(p) = (-ie)(ie)\int \frac{d^4p'}{(2\pi)^4} i\boldsymbol{D}^{\mu\nu}_{21}(k)(p+p')_\nu i\mathbf{\Delta}_{12}(p')(p+p')_\mu\,, \tag{34}$$

where we have defined

$$k = p' - p\,. \tag{35}$$

Substituting the photon and scalar propagators into this expression and using Eq. (27) we obtain

$$\begin{aligned} \mathrm{Im}\,\Pi_0(p) &= -\frac{e^2}{2}(2\pi)^2\epsilon(p\cdot u)\int \frac{d^4p'}{(2\pi)^4}\delta(k^2)\delta(p'^2-m^2)\epsilon(p'\cdot u)\epsilon(k\cdot u) \\ &\quad \times(-S^{\mu\nu})(p+p')_\mu(p+p')_\nu(f_B(k,0)-f_B(p',\alpha))\,, \end{aligned} \tag{36}$$

where the identity

$$f_B(-k,0)f_B(p',\alpha) = f_B(p,\alpha)[f_B(p',\alpha) - f_B(k,0)] \tag{37}$$

has been used. Eq. (36) can be written in the form

$$\begin{aligned} \mathrm{Im}\,\Pi_0(p) &= -\frac{e^2}{2}\epsilon(p\cdot u)\int \frac{d^3p'}{(2\pi)^3 2E'}\frac{d^3k}{(2\pi)^3 2\omega}(2\pi)^4(-S^{\mu\nu}) \\ &\quad \times\{\, \delta^{(4)}(p+k-p')(p+p')_\mu(p+p')_\nu[f_\gamma - f_\phi] \\ &\quad\; +\delta^{(4)}(p-k+p')(p-p')_\mu(p-p')_\nu[\overline{f}_\phi - f_\gamma] \\ &\quad\; +\delta^{(4)}(p-k-p')(p+p')_\mu(p+p')_\nu[1+f_\gamma+f_\phi] \\ &\quad\; +\delta^{(4)}(p+k+p')(p-p')_\mu(p-p')_\nu[1+f_\gamma+\overline{f}_\phi]\,\}\,, \end{aligned} \tag{38}$$

where, for the sake of brevity, we have called the photon and ϕ particle distributions by

$$\begin{aligned} f_\gamma &= f_B(k,0)\,, \\ f_\phi &= f_B(p',\alpha)\,, \end{aligned} \tag{39}$$

while $\overline{f}_\phi$ is the antiparticle density distribution, which is obtained from f_ϕ by changing the sign of α. In addition,

$$p'^\mu = (E',\vec{p}\,')\,, \qquad E' = \sqrt{\vec{p}\,'^2 + m^2}\,, \tag{40}$$

$$k^\mu = (\omega,\vec{k})\,, \qquad \omega = |\vec{k}|\,. \tag{41}$$

A more physically intuitive representation of this formula can be obtained by using Eq. (32) for $S_{\mu\nu}$. Then, introducing the amplitudes

$$M_A = (-ie)(p+p')^\mu \epsilon_\mu(k,\lambda)\,, \tag{42}$$

$$M_B = (-ie)(p-p')^\mu \epsilon_\mu(k,\lambda)\,, \tag{43}$$

we get

$$\mathrm{Im}\,\Pi_0 = -\,|p\cdot u|\,\Gamma(p)\,, \tag{44}$$

where we have defined

$$\begin{aligned} \Gamma(p) \equiv\ & \frac{1}{2p\cdot u}\int \frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3p'}{(2\pi)^3 2E'}(2\pi)^4 \\ & \times \{\ \delta^{(4)}(p+k-p')[f_\gamma(1+f_\phi) - f_\phi(1+f_\gamma)]\sum_{\lambda=1,2}|M_A|^2 \\ & \quad +\delta^{(4)}(p-k+p')[\overline{f}_\phi(1+f_\gamma) - f_\gamma(1+\overline{f}_\phi)]\sum_{\lambda=1,2}|M_B|^2 \\ & \quad +\delta^{(4)}(p-k-p')[(1+f_\gamma)(1+f_\phi) - f_\gamma f_\phi]\sum_{\lambda=1,2}|M_A|^2 \\ & \quad +\delta^{(4)}(p+k+p')[f_\gamma\overline{f}_\phi - (1+f_\gamma)(1+\overline{f}_\phi)]\sum_{\lambda=1,2}|M_B|^2\ \}\,. \end{aligned} \tag{45}$$

The formula for Γ given in Eq. (45) is immediately recognized as the total rate for a ϕ particle of energy p^0 and momentum $\vec{p}$ (as seen from the rest frame of the medium) with integrations over the phase space weighted by the statistical factors appropriate for each process [15]. M_A is the amplitude

for $\gamma\phi \to \phi$ or the decay $\phi \to \gamma\phi$, while M_B is the amplitude for $\phi\overline{\phi} \to \gamma$ or $\gamma\phi\overline{\phi} \to 0$. The amplitudes for the inverse reactions are given by the complex conjugates of M_A and M_B. For certain specific values of p^0 and $\vec{p}$ some of these processes will be kinematically forbidden, but in general Γ is non-zero.

In conclusion, the one-loop photon self-energy calculated with the full ϕ propagator given in Eq. (20) instead of the free propagator of Eq. (16), is defined at zero momentum. In particular, the absorptive part of the ϕ self-energy, which physically is related to the damping rate of the particle, cannot be neglected if the photon self-energy is evaluated at zero momentum. Then, the physical picture that emerges is the following. The traditional formulas that are given for

$$\lim_{|\vec{p}| \to 0} \pi_{\mu\nu}(0, \vec{p}) \tag{46}$$

and

$$\lim_{p^0 \to 0} \pi_{\mu\nu}(p^0, \vec{0}) \,, \tag{47}$$

which are related to well known physical quantities such as the plasma frequency and Debye radius, are valid in the limiting cases

$$\begin{aligned} p^0 = 0 \quad &; \quad \Gamma \ll |\vec{p}| \ll m \\ \vec{p} = 0 \quad &; \quad \Gamma \ll p^0 \ll m \,. \end{aligned} \tag{48}$$

Since the two limits correspond to two different physical situations the results are different. Traditionally Γ is omitted in the above conditions, but then it must be kept in mind that the formulas cannot be taken literally all the way to zero momentum. The same conclusion can be reached for fermionic QED and other field theories, which will be discussed in detail in a future publication.

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Figure captions

1. Photon self-energy diagrams.

2. Self-energy diagrams for the scalar.